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Raising and Lowering of Generalized Hulthén Potential from Supersymmetry Approaches

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We obtain the exact bound states of the generalized of Hulthén potential with negative energy levels using an analytic approach. In order to obtain bound states, we use the associated Jacobi differential equation. Using the supersymmetry approach to quantum mechanics, we show that these bound states, via four pairs of first order differential operators, represent four types of ladder equations. Two types of these supersymmetric structures suggest derivation of algebric solutions for the bound states using two different approaches.

KEY WORDS: special functions; supersymmetry; bound states; ladder operators; Hulthén potential.

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Supersymmetry in quantum mechanics is based on the concept of factorization in the context of shape invariant quantum mechanical problems. If a quantum mechanical problem possesses supersymmetry, we can then factorize the Hamiltonian of the system in terms of a product of first order differential operators leading to shape invariant equations. In this approach, the Hamiltonian is decomposed once in terms of the product of raising and lowering operators and once again as the product of lowering and raising operators, in such a way that the corresponding quantum states of successive levels, are their eigen-states . These Hamiltonians are called supersymmetric partner of each other. In fact, the three separate topics, the factorization method, supersymmetry in quantum mechanics and shape invariance, nowadays have converged to each other. Initially the factorization method was suggested by Darboux (1996) and Schrödinger (1940, 1941a, b) applied it to quantum mechanics. Infeld and Hull (1951) in their review article have studied a large variety of second order differential equations

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with boundary conditions and have classified them using six different types of factorizations. The idea of supersymmetry in the context of quantum mechanics were first studied by Nicolai and Witten and later by Cooper *et al.* (Beckers *et al.*, 1997; Cooper and Freedman, 1983; Nicolai, 1976; Witten, 1981, 1982). Later, Gendenshtein put forward the concept of shape invariance in the context of the supersymmetric quantum mechanics (Gendenshtein, 1983). Many features of shape invariant quantum mechanical problems have been studied using the factorization method in the context of supersymmetric quantum mechanics (Adrianov *et al.*, 1984; Aoyama *et al.*, 2001; Balantekin, 1998; Balantekin *et al.*, 1999; Barcelos-Neto and Das, 1986; Carinena and Ramos, 2000a, 2000b; Chuan, 1991; Cooper *et al.*, 1995, 2001; Das and Huang, 1990; Dunne and Feinberg, 1998; Dutt *et al.*, 1988; Fakhri, 2003; Fakhri and Seyed Yaghoobi, 2001; Fukui and Aizawa, 1993; Gendenshtein, 1983; Haymaker and Rau, 1986; Qian *et al.*, 2002; Salamonson and van Holten, 1982; Sukumar, 1985). Nowadays, the concept of shape invariance have been extended to the ordinary differential equations and on this basis a second order differential operator is decomposed to a product ladder operators (Jafarizadeh and Fakhri, 1997, 1998). The Hulthén potential (1925) is one of the important short - range potential in physics. This potential is a special case of the Eckart potential which has been widely used in several branches of physics, for example in nuclear (Hall, 1935) and atomic physics (Myhrman, 1983). From Ref (Lopez-Bonilla *et al.*, 2002) the generalized Hulthén potential is given by:

$$V(r) = \frac{V_0}{1 - e^{Ar}} + \frac{2A^2 e^{Ar}}{(1 - e^{Ar})^2}$$

where A and V_0 are positive constants such that $V_0 > A^2$ and A is called screening parameter.

In this paper, we will exactly solve the time-independent Schrödinger equation for the generalized spherical Hulthén potential with the zero angular momentum,

$$\frac{-\hbar^2}{2M}\left(\frac{d^2\psi(r)}{dr^2} + \frac{2}{r}\frac{d\psi(r)}{dr}\right) + \left(\frac{V_0}{1 - e^{Ar}} + \frac{2A^2 e^{Ar}}{(1 - e^{Ar})^2}\right)\psi(r) = E\psi(r), \quad (1)$$

and apply supersymmetry factorization approaches to it. We compute the parameters V_0, E and also the bound states $\psi(r)$ from the comparison of the differential equation (1) with the associated Jacobi differential equation in an appropriate manner. To begin with, we recall that for the real parameters $\alpha, \beta > -1$, the associated Jacobi differential equation corresponding to $P_{n,m}^{(\alpha,\beta)}(x)$ in the interval $x \in (-1, 1)$

is given as follows (Jafarizadeh and Fakhri, 1997, 1998):

$$(1-x^2)P''^{(\alpha,\beta)}_{n,m}(x) - [\alpha-\beta+(\alpha+\beta+2)x]\,P'^{(\alpha,\beta)}_{n,m}(x)$$
$$+\left[n(\alpha+\beta+n+1) - \frac{m(\alpha+\beta+m+(\alpha-\beta)x)}{1-x^2}\right]P^{(\alpha,\beta)}_{n,m}(x) = 0\,. \tag{2}$$

Here, the indices n and m are non-negative integers with $\infty > n \geq m \geq 0$, and for $m = 0$, the Eq. (2) reduces to the differential equation leading to the Jacobi polynomials. The associated Jacobi functions $P^{(\alpha,\beta)}_{n,m}(x)$ are solutions to the differential Eq. (2) and have the following Rodrigues representation

$$P^{(\alpha,\beta)}_{n,m}(x) = \frac{a_{n,m}(\alpha,\beta)}{(1-x)^{\alpha+\frac{m}{2}}(1+x)^{\beta+\frac{m}{2}}}\left(\frac{d}{dx}\right)^{n-m}((1-x)^{\alpha+n}(1+x)^{\beta+n})\,. \tag{3}$$

As mentioned in (Carinena, 2000a, 200b), we can write the associated Jacobi differential Eq. (2) as the following shape invariant equations with respect to the parameters n and m:

$$A^{+}_{n,m}(x)A^{-}_{n,m}(x)P^{(\alpha,\beta)}_{n,m}(x) = E_{n,m}P^{(\alpha,\beta)}_{n,m}(x) \tag{4a}$$

$$A^{-}_{n,m}(x)A^{+}_{n,m}(x)P^{(\alpha,\beta)}_{n-1,m}(x) = E_{n,m}P^{(\alpha,\beta)}_{n-1,m}(x), \tag{4b}$$

and

$$A^{+}_{m}(x)A^{-}_{m}(x)P^{(\alpha,\beta)}_{n,m}(x) = \mathcal{E}_{n,m}P^{(\alpha,\beta)}_{n,m}(x) \tag{5a}$$

$$A^{-}_{m}(x)A^{+}_{m}(x)P^{(\alpha,\beta)}_{n,m-1}(x) = \mathcal{E}_{n,m}P^{(\alpha,\beta)}_{n,m-1}(x), \tag{5b}$$

where

$$A^{+}_{n,m}(x) = (1-x^2)\frac{d}{dx} - (\alpha+\beta+n)x - \frac{(\alpha-\beta)(\alpha+\beta+n+m)}{\alpha+\beta+2n}$$
$$A^{-}_{n,m}(x) = -(1-x^2)\frac{d}{dx} - nx + \frac{(\alpha-\beta)(n-m)}{\alpha+\beta+2n}, \tag{6}$$

$$A^{+}_{m}(x) = \sqrt{1-x^2}\frac{d}{dx} + \frac{(m-1)x}{\sqrt{1-x^2}}$$
$$A^{-}_{m}(x) = -\sqrt{1-x^2}\frac{d}{dx} + \frac{(\alpha-\beta)+(\alpha+\beta+m)x}{\sqrt{1-x^2}}, \tag{7}$$

$$E_{n,m} = \frac{4(n-m)(\alpha+n)(\beta+n)(\alpha+\beta+n+m)}{(\alpha+\beta+2n)^2}, \tag{8}$$

and

$$\mathcal{E}_{n,m} = (n-m+1)(\alpha+\beta+n+m). \tag{9}$$

Note that the shape invariant Eq. (4) contain the indices (n, m) and also $(n-1, m)$ and the shape invariant Eq. (5) contain the indices (n, m) and $(n, m-1)$. Using Eqs. (6) to (9) it is not difficult to see that each of the Eqs. (4a), (4b), (5a) and (5b) is a copy of the differential Eq. (2). The facotrized Eqs. (4a) and (4b) together describe shape invariance with respect to n, and also (5a) and (5b) together describe shape invariance with respect to m. One can easily write the shape invariant Eqs. (4) and (5) as the laddering relations with respect to the indices n and m, respectively:

$$A^{+}_{n,m}(x)P^{(\alpha,\beta)}_{n-1,m}(x) = \sqrt{E_{n,m}}\,P^{(\alpha,\beta)}_{n,m}(x) \tag{10a}$$

$$A^{-}_{n,m}(x)P^{(\alpha,\beta)}_{n,m}(x) = \sqrt{E_{n,m}}\,P^{(\alpha,\beta)}_{n-1,m}(x) \tag{10b}$$

$$A^{+}_{m}(x)P^{(\alpha,\beta)}_{n,m-1}(x) = \sqrt{\mathcal{E}_{n,m}}\,P^{(\alpha,\beta)}_{n,m}(x) \tag{11a}$$

$$A^{-}_{m}(x)P^{(\alpha,\beta)}_{n,m}(x) = \sqrt{\mathcal{E}_{n,m}}\,P^{(\alpha,\beta)}_{n,m-1}(x). \tag{11b}$$

It is clear that in contrast to Eqs. (4) and (5), the realization of Eqs. (10) and (11) imposes a constraint on the normalization coefficients $a_{n,m}(\alpha, \beta)$. A simple computation shows that with the choice of the following normalization coefficient

$$a_{n,m}(\alpha, \beta) = \frac{(-1)^m}{2^n}\sqrt{\frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(n-m+1)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}}\,C(\alpha, \beta) \quad n \geq m, \tag{12}$$

with the constraint

$$C(\alpha+m, \beta+m) = \left(\frac{-1}{2}\right)^m C(\alpha, \beta), \tag{13}$$

Eqs. (10) and (11) can be simultaneously satisfied. The arbitrary constant $C(\alpha, \beta)$ is independent of n and m and satisfies only the constraint (13). Now, with the application of Eq. (12) and using integration by part, for the given associated Jacobi function $P^{(\alpha,\beta)}_{n,m}(x)$ in Eq. (3), we obtain the following orthogonality condition

$$\int_{-1}^{1} P^{(\alpha,\beta)}_{n,m}(x)P^{(\alpha,\beta)}_{n',m}(x)(1-x)^{\alpha}(1+x)^{\beta}dx = \delta_{nn'}h^2_n(\alpha, \beta). \tag{14}$$

The above equation shows that $h^2_n(\alpha, \beta)$ is the squared norm of the associated Jacobi function $P^{(\alpha,\beta)}_{n,m}(x)$ which is also independent of m. This in turn, leads to following equation:

$$h^2_n(\alpha, \beta) = \frac{2^{\alpha+\beta+1}}{\alpha+\beta+2n+1}\,C^2(\alpha, \beta). \tag{15}$$

If we know that $C(\alpha, \beta)$ is symmetric with respect to the exchange of parameters α and β, namely $C(\alpha, \beta) = C(\beta, \alpha)$ then according to Eq. (12), also normalization coefficients $a_{n,m}(\alpha, \beta)$ will be symmetric with under the exchange of α and β, namely $a_{n,m}(\alpha, \beta) = a_{n,m}(\beta, \alpha)$. Hence, from Eq. (3) one can obtain $P_{n,m}^{(\alpha,\beta)}(-x) = (-1)^{n-m} P_{n,m}^{(\beta,\alpha)}(x)$. So from (14) we have

$$\int_0^1 P_{n,m}^{(\alpha,\beta)}(x) P_{n',m}^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta} dx + (-1)^{n+n'} \int_0^1 P_{n,m}^{(\beta,\alpha)}(x) P_{n',m}^{(\beta,\alpha)}(x)(1-x)^{\beta}(1+x)^{\alpha} dx = \delta_{nn'} h_n^2(\alpha, \beta). \quad (16)$$

The change of variable

$$x = \coth \frac{Ar}{2} \quad (17)$$

with the real parameter $A > 0$ transforms the interval $x \in (0, 1)$ in Eq. (16) to the interval $r \in (0, \infty)$. By change of function

$$\psi(r(x)) = u(x) P_{n,m}^{(\alpha,\beta)}(x) \quad \text{with} \quad u(x) = \frac{(1-x^2)^{\frac{\alpha+\beta}{4}}}{\coth^{-1} x} e^{\frac{1}{2}(\beta-\alpha)\tanh^{-1} x}, \quad (18)$$

together with the change of variable (17) in Eq. (1), it is easy to obtain the differential Eq. (2) for $P_{n,m}^{(\alpha,\beta)}(x)$. Consequently, comparing Eqs. (1) and (2) and using the change of variable (17) together with the change of function (18), we obtain

$$V_0(\alpha, \beta; m) = \frac{\hbar^2}{8MA^2}(\beta - \alpha)(\alpha + \beta + 2m) \quad (19)$$

and

$$\psi(r(x)) = \frac{(1-x^2)^{\frac{\alpha+\beta}{4}}}{\coth^{-1} x} e^{\frac{1}{2}(\beta-\alpha)\tanh^{-1} x} P_{n,m}^{(\alpha,\beta)}(x) \quad (20)$$

and also we can obtain the Schrödinger equation corresponding to the generalized Hulthén potential with the zero angular momentum:

$$\left[\frac{-\hbar^2}{2M}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{r}\right) + \left(\frac{V_0(\alpha, \beta; m)}{1 - e^{Ar}} + \frac{2A^2(\alpha, \beta; m) e^{Ar}}{(1 - e^{Ar})^2}\right)\right] \left|{}^{\alpha,\beta}_{n,m}\right\rangle = E_m(\alpha) \left|{}^{\alpha,\beta}_{n,m}\right\rangle. \quad (21)$$

The bound states and it's energy spectrum are given as follows

$$\left|{}^{\alpha,\beta}_{n,m}\right\rangle = \frac{2}{Ah_n(\alpha, \beta)} \frac{e^{\frac{1}{2}(\beta-\alpha)\tanh^{-1}(\coth \frac{Ar}{2})} P_{n,m}^{(\alpha,\beta)}(\coth \frac{Ar}{2})}{r \sinh^{\frac{\alpha+\beta}{2}}(\frac{Ar}{2})} \quad n \geq m, \quad (22)$$

$$E_m(\alpha) = \frac{-\hbar^2}{8Ma^2}(\alpha + m)^2. \quad (23)$$

It is also necessary to mention that, in special the case $\alpha = \beta$, the first term of the potential is cancelled, $V_0(\alpha, \alpha; m) = 0$, in this case the second term of potential becomes independent of the parameter m. In this case, m plays role of a quantum number and the quantum of energy proportional to minus $(\alpha + m)^2$. And also it is clear that for a given m, all bound states $|{}^{\alpha,\beta}_{n,m}\rangle$ with $n \geq m$, have the same binding energy $E_m(\alpha)$. For usual Spherical Hulthén potential, bound states are expressed in terms of the hypergeometric functions. Here we indeed deal with such a case. The reader can check for the relation-ship between the associated hypergeometric functions and associated Jacobi functions in Fernández *et al.* (1996). Using the definition of the inner product of the bound states $|{}^{\alpha,\beta}_{n,m}\rangle$ with respect to the weight function

$$W(r) = \frac{r^2 A^3 \left(1 - \coth \frac{Ar}{2}\right)^{\alpha} \left(1 + \coth \frac{Ar}{2}\right)^{\beta} \sinh^{\alpha+\beta-2} \frac{Ar}{2}}{8 e^{(\beta-\alpha)\tanh^{-1}(\coth \frac{Ar}{2})}}, \tag{24}$$

in the interval $r \in (0, \infty)$ by

$$\begin{aligned}\left\langle{}^{\alpha,\beta}_{n,m}\middle|{}^{\alpha,\beta}_{n',m}\right\rangle = {} & \frac{A}{2h_n(\alpha, \beta) h_{n'}(\alpha, \beta)} \\ & \times \int_{r=0}^{\infty} P^{(\alpha,\beta)}_{n,m}\left(\coth \frac{Ar}{2}\right) P^{(\alpha,\beta)}_{n',m}\left(\coth \frac{Ar}{2}\right)\left(1 - \coth \frac{Ar}{2}\right)^{\alpha} \\ & \times \left(1 + \coth \frac{Ar}{2}\right)^{\beta} \frac{dr}{\sinh^2 \frac{Ar}{2}},\end{aligned} \tag{25}$$

in Eq. (16) we obtain

$$\left\langle{}^{\alpha,\beta}_{n,m}\middle|{}^{\alpha,\beta}_{n',m}\right\rangle + (-1)^{n+n'} \left\langle{}^{\beta,\alpha}_{n,m}\middle|{}^{\beta,\alpha}_{n',m}\right\rangle = \delta_{nn'}. \tag{26}$$

The existence of each pair of the raising and lowering operators provides us with two fermionic operators and one bosonic operator, which together satisfy $\mathcal{N} = 2$ supersymmetry algebra . Therefore, the representation superspace of the supersymmetry algebra is written as a direct sum of the fermionic and bosonic sectors which are constructed by the bound states of the generalaized Hulthén potential. So, in order to obtain different representations of the $\mathcal{N} = 2$ supersymmetry algebra , it is enough to obtain the different representations of the ladder relations for bound states of the generalized Hulthén potential.

Now, we can obtain four pairs of ladder relations for the bound states of the generalized Hulthén potential, $|{}^{\alpha,\beta}_{n,m}\rangle$. First of all we show that these bound states represent the raising and lowering relations with respect to the index n as a first approach to supersymmetry. Defining the operators

$$A^{\pm}_{n,m}(r) := [u(x) A^{\pm}_{n,m}(x) u^{-1}(x)]_{x=\coth(\frac{Ar}{2})}, \tag{27}$$

and using the explicit form of $u(x)$, we can compute their differential forms as follows

$$A^{\pm}_{n,m}(r) = \pm\frac{2}{A}\frac{d}{dr} - \frac{1}{2}(\alpha+\beta+2n)\coth\frac{Ar}{2} \pm \frac{2}{Ar} - \frac{(\alpha-\beta)(\alpha+\beta+2m)}{2(\alpha+\beta+2n)}. \tag{28}$$

Consequently, by using Eqs. (10) and (22) we conclude that the bound states $|{}^{\alpha,\beta}_{n,m}\rangle$ represent the raising and lowering relations of the index n as follows

$$A^{+}_{n,m}(r)\left|{}^{\alpha,\beta}_{n-1,m}\right\rangle = \frac{h_n(\alpha,\beta)}{h_{n-1}(\alpha,\beta)}\sqrt{E_{n,m}}\,\left|{}^{\alpha,\beta}_{n,m}\right\rangle \tag{29a}$$

$$A^{-}_{n,m}(r)\left|{}^{\alpha,\beta}_{n,m}\right\rangle = \frac{h_{n-1}(\alpha,\beta)}{h_n(\alpha,\beta)}\sqrt{E_{n,m}}\,\left|{}^{\alpha,\beta}_{n-1,m}\right\rangle. \tag{29b}$$

Combination of Eqs. (29a) and (29b) with each other, leads to shape invariant equations with respect to the index n as (n, m) and $(n-1, m)$ which is represented by the bound states of the generalized Hulthén potential.

Now we are going to investigate a second approach to supersymmetry, namely representation of raising and lowering relations with respect to the index m using the bound states of the generalized Hulthén potential. The explicit form of the first order differential operators are defined as follows

$$A^{\pm}_{m}(r) := \left[u(x)A^{\pm}_{m}(x)u^{-1}(x)\right]_{x=\coth\frac{Ar}{2}}. \tag{30}$$

Using Eq. (7) and also given $u(x)$ as in (18), are obtained as follows

$$\begin{aligned} A^{\pm}_{m}(r) = &\pm\frac{2}{A}\sinh\frac{Ar}{2}\frac{d}{dr} + \frac{1}{2}(\alpha+\beta+2m-1\mp 1)\cosh\frac{Ar}{2} \\ &+ \frac{1}{2}(\alpha-\beta)\sinh\frac{Ar}{2} \pm \frac{2}{Ar}\sinh\frac{Ar}{2}. \end{aligned} \tag{31}$$

From Eqs. (11) and (22) we can easily obtain that the operators $A^{+}_{m}(r)$ and $A^{-}_{m}(r)$ increase and decrease the index m respectively as the following

$$A^{+}_{m}(r)\left|{}^{\alpha,\beta}_{n,m-1}\right\rangle = \sqrt{\mathcal{E}_{n,m}}\,\left|{}^{\alpha,\beta}_{n,m}\right\rangle \tag{32a}$$

$$A^{-}_{m}(r)\left|{}^{\alpha,\beta}_{n,m}\right\rangle = \sqrt{\mathcal{E}_{n,m}}\,\left|{}^{\alpha,\beta}_{n,m-1}\right\rangle. \tag{32b}$$

Again, combination of Eqs. (32a) and (32b) with each other, gives us shape invariant equations with respect to the index m as (n, m) and $(n, m-1)$. These are represented by the bound states of the generalized Hulthén potential.

Now we focus on a third approach to supersymmetry, this follows from different type of the laddering relations. Our purpose of laddering relations here is to introduce first order differential operators which shift simultaneously the indices n and m of the bound states by one unit, so that one of them increases while other decreases. The new first order differential operators are defined as below

$$A^{+,-}_{n,m}(r) := A^{-}_{m}(r)A^{+}_{n,m}(r) - A^{+}_{n,m-1}(r)A^{-}_{m}(r)$$
$$A^{-,+}_{n,m}(r) := A^{-}_{n,m}(r)A^{+}_{m}(r) - A^{+}_{m}(r)A^{-}_{n,m-1}(r). \tag{33}$$

In turns out that $A^{\pm,\pm}_{n,m}(r)$ have the following explicit form

$$\begin{aligned} A^{\pm,\mp}_{n,m}(r) = & \left[\frac{\alpha-\beta}{\alpha+\beta+2n} + \coth\frac{Ar}{2}\right]\left[\pm\frac{2}{A}\sinh\frac{Ar}{2}\frac{d}{dr} - \frac{1}{2}(\alpha-\beta)\sinh\frac{Ar}{2}\right. \\ & \left. -\frac{1}{2}(\alpha+\beta+2m-1\pm 1)\cosh\frac{Ar}{2} \pm \frac{2}{Ar}\sinh\frac{Ar}{2}\right] \\ & + \left(n-m+\frac{1}{2}\mp\frac{1}{2}\right)\frac{1}{\sinh\frac{Ar}{2}}. \end{aligned} \tag{34}$$

By using Eqs. (29) and (32) we obtain the following new laddering relations

$$A^{+,-}_{n,m}(r)\left|{}^{\alpha,\beta}_{n-1,m}\right\rangle = \frac{2h_n(\alpha,\beta)}{h_{n-1}(\alpha,\beta)}\sqrt{\frac{(n-m+1)(n-m)(\alpha+n)(\beta+n)}{(\alpha+\beta+2n)^2}}\left|{}^{\alpha,\beta}_{n,m-1}\right\rangle \tag{35a}$$

$$A^{-,+}_{n,m}(r)\left|{}^{\alpha,\beta}_{n,m-1}\right\rangle = \frac{2h_{n-1}(\alpha,\beta)}{h_n(\alpha,\beta)}\sqrt{\frac{(n-m+1)(n-m)(\alpha+n)(\beta+n)}{(\alpha+\beta+2n)^2}}\left|{}^{\alpha,\beta}_{n-1,m}\right\rangle. \tag{35b}$$

The laddering relations (35) show that the operator $A^{+,-}_{n,m}(r)(A^{-,+}_{n,m}(r))$ increases and decreases (decreases and increases) simultaneously the indices n and m of bound states of the generalized Hulthén potential. Notice that, the combination of two laddering Eqs. (35a) and (35b) with each other, gives us shape invariant equations with respect to both indices n and m as $(n, m-1)$ and $(n-1, m)$, which are realized by the bound states of the generalized Hulthén potential.

A fourth approaches to supersymmetry is obtained from the realization of the laddering relations simultaneously for raising and lowering operators of both indices n and m of the generalized Hulthén potential bound states. Defining the new first order differential operators as

$$\begin{aligned} A^{+,+}_{n,m}(r) &:= A^{+}_{m}(r)A^{+}_{n,m-1}(r) - A^{+}_{n,m}(r)A^{+}_{m}(r) \\ A^{-,-}_{n,m}(r) &:= A^{-}_{n,m-1}(r)A^{-}_{m}(r) - A^{-}_{m}(r)A^{-}_{n,m}(r). \end{aligned} \tag{36}$$

So, using Eqs. (28) and (31) one may obtain the explicit form of these operators as follows

$$A_{n,m}^{\pm,\pm}(r) = \left[\frac{\alpha-\beta}{\alpha+\beta+2n} - \coth\frac{Ar}{2}\right]\left[\pm\frac{2}{A}\sinh\frac{Ar}{2}\frac{d}{dr} + \frac{1}{2}(\alpha-\beta)\sinh\frac{Ar}{2}\right.$$
$$\left. + \frac{1}{2}(\alpha+\beta+2m-1\mp 1)\cosh\frac{Ar}{2} \pm \frac{2}{Ar}\sinh\frac{Ar}{2}\right]$$
$$-\left(\alpha+\beta+n+m-\frac{1}{2}\mp\frac{1}{2}\right)\frac{1}{\sinh\frac{Ar}{2}}. \tag{37}$$

For these operators by applying Eq. (36), and from the scalar relations (29) and (32), immediately we obtain the following new laddering relations

$$A_{n,m}^{+,+}(r)\left|{}^{\alpha,\beta}_{n-1,m-1}\right\rangle$$
$$= \frac{2h_n(\alpha,\beta)}{h_{n-1}(\alpha,\beta)}\sqrt{\frac{(\alpha+\beta+n+m-1)(\alpha+\beta+n+m)(\alpha+n)(\beta+n)}{(\alpha+\beta+2n)^2}}$$
$$\times\left|{}^{\alpha,\beta}_{n,m}\right\rangle \tag{38a}$$

$$A_{n,m}^{-,-}(r)\left|{}^{\alpha,\beta}_{n,m}\right\rangle$$
$$= \frac{2h_{n-1}(\alpha,\beta)}{h_n(\alpha,\beta)}\sqrt{\frac{(\alpha+\beta+n+m-1)(\alpha+\beta+n+m)(\alpha+n)(\beta+n)}{(\alpha+\beta+2n)^2}}$$
$$\times\left|{}^{\alpha,\beta}_{n-1,m-1}\right\rangle. \tag{38b}$$

Therefore, the operators $A_{n,m}^{+,+}(r)$ and $A_{n,m}^{-,-}(r)$ increase and decrease simultaneously both the indices n and m of bound states, respectively. Combination of the laddering Eqs. (38a) and (38b) with each other, rise to shape invariant equations with respect to the indices n and m as (n, m) and $(n-1, m-1)$.

In order to obtain two types of different algebraic solutions for the bound states of the generalized Hulthén potential, we now use the supersymmetry approaches of the first and second types, i.e. the laddering Eqs. (29) and (32). First, we obtain the suggested algebraic solution by the laddering equations with respect to the index n. Pay attention for a given m, and using $E_{m,m} = 0$, Eq. (29b) gives

$$A_{m,m}^{-}(r)\left|{}^{\alpha,\beta}_{m,m}\right\rangle = 0\,, \tag{39}$$

which is a first order differential equation and its solution is easily obtained as follows

$$\left|{}^{\alpha,\beta}_{m,m}\right\rangle = \frac{2\,a_{m,m}(\alpha,\beta)}{Ah_m(\alpha,\beta)}\frac{e^{\frac{1}{2}(\beta-\alpha)\tanh^{-1}(\coth\frac{Ar}{2})}}{r\sinh^{\frac{\alpha+\beta+2m}{2}}\frac{Ar}{2}}. \tag{40}$$

Also, the result (40) can be confirmed from the analytic solution (22). With successive application of laddering Eq. (29a), all other bound states $|^{\alpha,\beta}_{n,m}\rangle$ with $n > m$ can be obtained,

$$\left|^{\alpha,\beta}_{n,m}\right\rangle = \frac{h_m(\alpha,\beta)}{h_n(\alpha,\beta)} \frac{A^+_{n,m}(r)A^+_{n-1,m}(r)\cdots A^+_{m+1,m}(r)\left|^{\alpha,\beta}_{m,m}\right\rangle}{\sqrt{E_{n,m}E_{n-1,m}\cdots E_{m+1,m}}} \qquad n \geq m+1. \tag{41}$$

It is clear that all bound states $|^{\alpha,\beta}_{n,m}\rangle$ with $n \geq m$ which is computed by algebraic methods (Eqs. (40) and (41)) have the same energy $E_m(\alpha)$ and shifting the index n by the operators $A^{\pm}_{n,m}(r)$ do not change their energy. Now we obtain the algebraic solution which is suggested by laddering equations with respect to the index m. For a given n, with the consideration of $\mathcal{E}_{n,n+1} = 0$ from Eq. (32a) we have

$$A^+_{n+1}(r)\left|^{\alpha,\beta}_{n,n}\right\rangle = 0, \tag{42}$$

which is a first order differential equation, and it's solution may easily be obtained as (40), with the m replaced n. The energy of the bound state $|^{\alpha,\beta}_{n,n}\rangle$ is $E_n(\alpha)$. With successive application of the laddering Eq. (32b) one can obtain

$$\left|^{\alpha,\beta}_{n,m}\right\rangle = \frac{A^-_{m+1}(r)A^-_{m+2}(r)\cdots A^-_{n}(r)\left|^{\alpha,\beta}_{n,n}\right\rangle}{\sqrt{\mathcal{E}_{n,m+1}\mathcal{E}_{n,m+2}\cdots\mathcal{E}_{n,n}}} \qquad m \leq n-1. \tag{43}$$

Contrary to the previous case, all obtained bound states of the form of the algebraic solution (43) do not contain the same energy and the corresponding energies are given by $E_m(\alpha)$ which differ by the values of m. So the effect of the operators $A^{\pm}_m(r)$ on the bound states of the generalized Hulthén potential is to change the binding energy. The absolute value of the binding energy increases by $A^+_m(r)$ and decreases by $A^-_m(r)$. One can verify that the operators $A^{\pm;\mp}_{n,m}(r)$ and $A^{\pm;\pm}_{n,m}(r)$ shift the binding energy.

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